Euclid and the Greatest Common Divisor

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A Futuristic Look Through Ancient Lenses:
A Symposium on Ancient Greece
October 29, 2012

The school of Athens

Raphael fresco (1509–1510)

Euclid (or Archimedes?) with students

Euclid as geometer

Oxford University celebrates the sciences

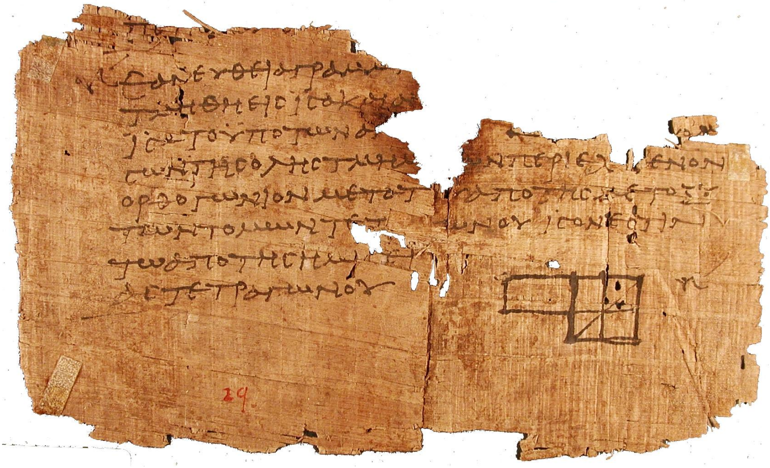
"Cathedral to Science" : 28 statues of scientists, philosophers, and engineers

Joseph Durham, Oxford University Museum of Natural History

What do we know about Euclid?

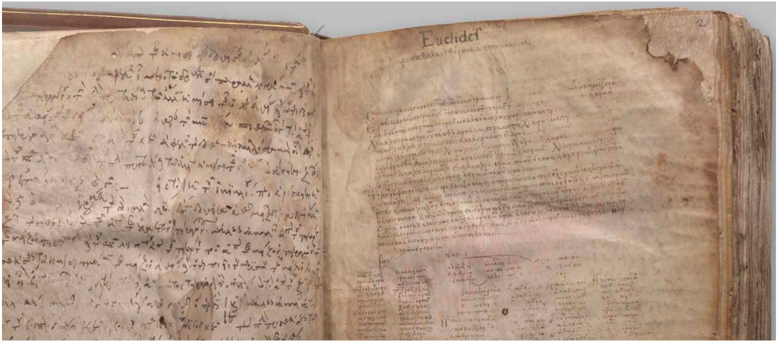
- Born ca. 300 BCE
- Prominent mathematician of antiquity — "father of geometry"
- Authored mathematical treatise Elements;
  foundation for logic, mathematics and modern science
- Taught at Alexandria during time of King Ptolemy I
- Provided rigorous foundation:
  - definitions
  - postulates (axioms)
  - proofs
- Author of other books, including Optics and Elements of Music
- Response to king (?): "There is no royal road to geometry."

Papyrus fragment: ca. 100 CE



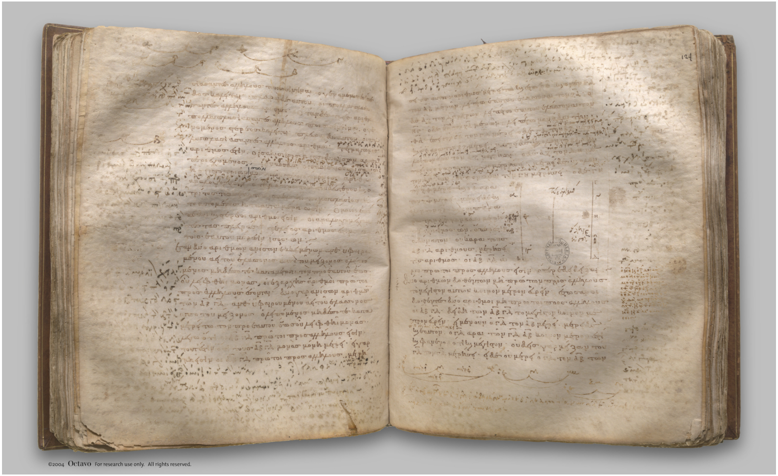
Bodleian library, Oxford: 888 CE

Title page



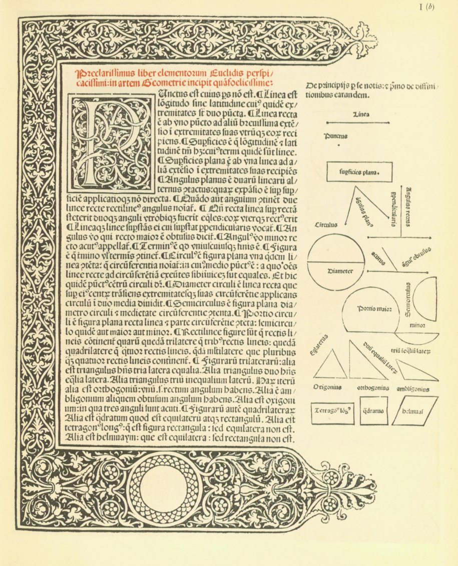
Bodleian library: 888 CE

Book VII, Proposition 1

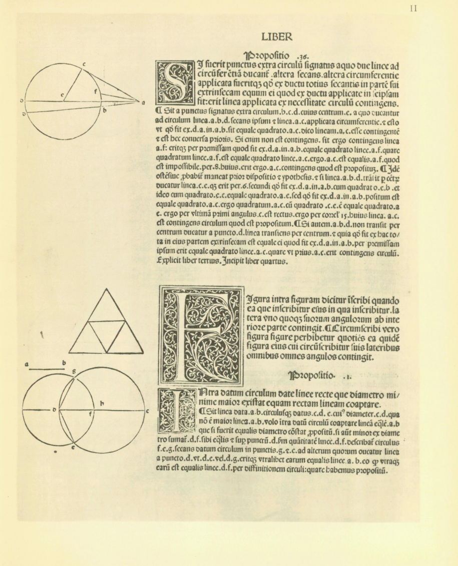


Elementa Geometriae, Venice: 1482

First typeset edition



Elementa Geometriae, Venice: 1482



11

LIBER

Propositio .36.

SI fuerit punctus extra circulū signatus aquo due linee ad circūferētiā ducant .altera secans.altera circumferentie applicata fueritq3 qđ ex ductu totius secantis in partē sui extrinsecam equum ei quod ex ductu applicate in seipsam fit:erit linea applicata ex necessitate circulū contingens. ¶ Sit a punctus signatus extra circulum.b.c.d.cuius centrum.e. a quo ducantur ad circulum linea.a.b.d.secans ipsum ⁊ linea.a.c.applicata circumferentie.⁊ esto vt qđ fit ex.d.a.in.a.b.sit equale quadrato.a.c.dico lineam.a.c.esse contingentē ⁊ est hec conuersa prioris. Si enim non est contingens. sit ergo contingens linea a.f: eritq3 per premissam quod fit ex.d.a.in.a.b.equale quadrato linee.a.f.quare quadratum linee.a.f.est equale quadrato linee.a.c.ergo.a.c.est equalis.a.f.quod est impossibile.per.8.huius.erit ergo.a.c.contingens quod est propositū. ¶ Idē ostēsiue ꝑbabiť maneat prior dispositio ⁊ ypothesis.⁊ si linea.a.b.d.trāsit ꝑ cētꝝ ducatur linea.c.e.q3 erit per.6.secundi qđ fit ex.d.a.in.a.b.cum quadrato.e.b .et ideo cum quadrato.c.e.equale quadrato.a.e.sed qđ fit ex.d.a.in.a.b.positum est equale quadrato.a.c.ergo quadratum.a.c.cū quadrato .c.e.ē equale quadrato.a e. ergo per vltimā primi angulus.c.est rectus.ergo per conel' 15.huius linea. a.c. est contingens circulum quod est propositum. ¶ Si autem.a.b.d.non transit per centrum ducatur a puncto.d.linea transiens per centrum.⁊ quia qđ fit ex hac tota in eius partem extrinsecam est equale ei quod fit ex.d.a.in.a.b.per premissam ipsum erit equale quadrato linee.a.c.quare vt prius.a.c.erit contingens circulū. Explicit liber tertius. Incipit liber quartus.

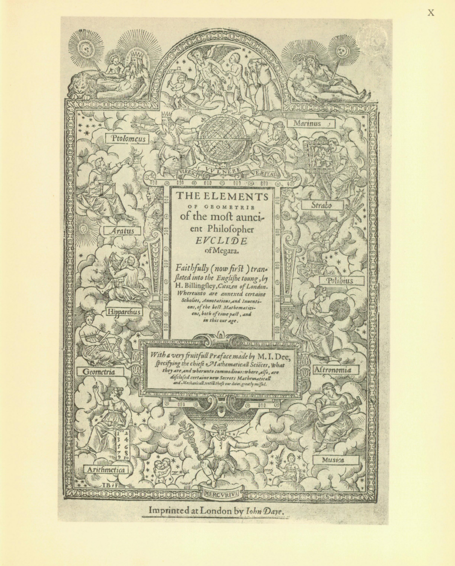
FIgura intra figuram dicitur īscribi quando ea que inscribitur eius in qua inscribitur.latera vno quoq3 suorum angulorum ab interiore parte contingit. ¶ Circumscribi vero figura figure perhibetur quotiēs ea quidē figura eius cui circūscribitur suis lateribus omnibus omnes angulos contingit.

Propositio. .1.

INtra datum circulum date linee recte que diametro minime maior existat equam rectam lineam coaptare. ¶ Sit linea data.a.b.circulusq3 datus.c.d. e.cui⁹ diameter.c.d.qua nō ē maior linea.a.b.volo itra datū circulū coaptare lineā eq̄lē.a.b que si fuerit equalis diametro cōstat ꝑpositū.si aūt minor ex diametro sumať.d.f.sibi eq̄lis ⁊ sup punctū.d.ſm quātitatē linee.d.f.describať circulus f.e.g.secans datum circulum in punctis.g.⁊.c.ad alterum quorum ducatur linea a puncto.d.vt.d.e.vel.d.g.eritq3 vtralibet earum equalis linee.a. b.eo ꝙ vtraq3 earū est equalis linee.d.f.per diffinitionem circuli:quare habemus propositū.

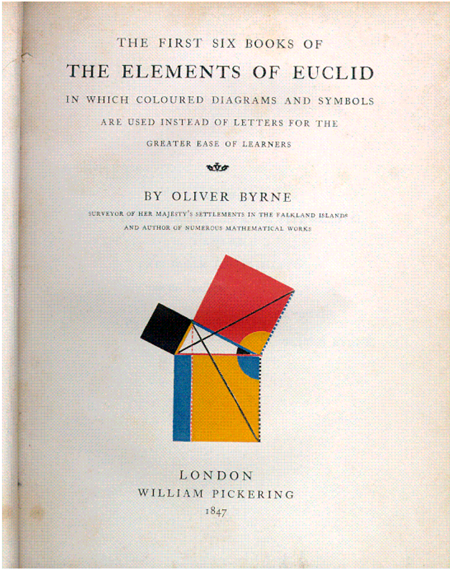
First English version

London: 1570



Oliver Byrne: 1847

Attempts to "color-code" mathematical proofs


THE FIRST SIX BOOKS OF

THE ELEMENTS OF EUCLID

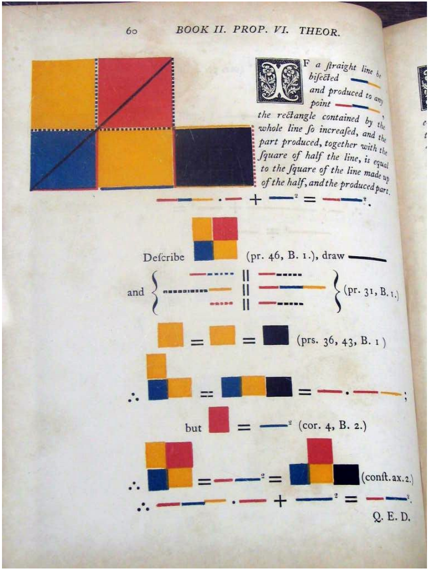
IN WHICH COLOURED DIAGRAMS AND SYMBOLS ARE USED INSTEAD OF LETTERS FOR THE GREATER EASE OF LEARNERS

BY OLIVER BYRNE

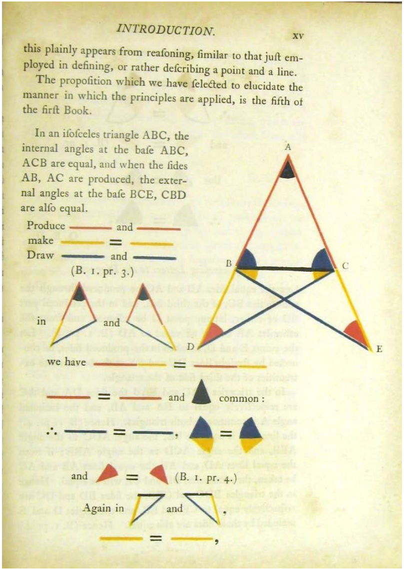
SURVEYOR OF HER MAJESTY'S SETTLEMENTS IN THE FALKLAND ISLANDS AND AUTHOR OF NUMEROUS MATHEMATICAL WORKS

LONDON
WILLIAM PICKERING
1847

Oliver Byrne: 1847



Oliver Byrne: 1847



INTRODUCTION. xv

this plainly appears from reaſoning, ſimilar to that juſt employed in defining, or rather deſcribing a point and a line.

The propoſition which we have ſelected to elucidate the manner in which the principles are applied, is the fifth of the firſt Book.

In an iſoſceles triangle ABC, the internal angles at the baſe ABC, ACB are equal, and when the ſides AB, AC are produced, the external angles at the baſe BCE, CBD are alſo equal.

Produce and

make =

Draw and

(B. 1. pr. 3.)

in and

we have =

= and common:

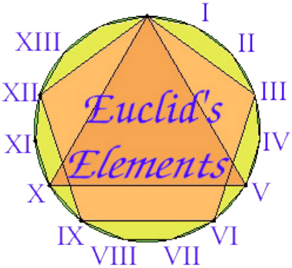
∴ = =

and = (B. 1. pr. 4.)

Again in and ,

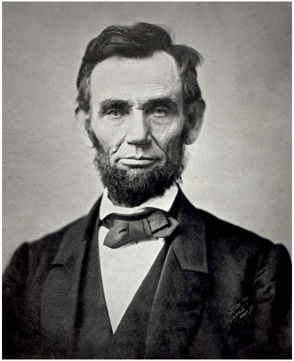
= ,

Online: with Java applets



http://aleph0.clarku.edu/~djoyce/java/elements/elements.html

The Abraham Lincoln connection



"He studied and nearly mastered the Six-books of Euclid (geometry) since he was a member of Congress. He began a course of rigid mental discipline with the intent to improve his faculties, especially his powers of logic and language. Hence his fondness for Euclid, which he carried with him on the circuit till he could demonstrate with ease all the propositions in the six books; often studying far into the night, with a candle near his pillow, while his fellow-lawyers, half a dozen in a room, filled the air with interminable snoring."

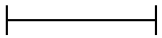
–Lincoln's law partner, William Herndon

Euclid: founder of number theory

- Number theory is the study of the integers
- Euclid introduced concepts of number theory:
  - whole number
  - prime number
  - composite number
  - perfect number
    (e.g., $6 = 1 + 2 + 3$, $28 = 1 + 2 + 4 + 7 + 14$)
- Major results
  - Method to find the greatest common divisor of two whole numbers
  - Whole numbers can be uniquely factored into primes
    (e.g., $1035 = 3 \cdot 3 \cdot 5 \cdot 23$ and this is unique)
  - There are an infinite number of primes
  - If $2^p - 1$ is prime, then $2^{p-1}(2^p - 1)$ is perfect
    (e.g., for $p = 3, 2^3 - 1$ is prime and so $2^2(2^3 - 1) = 28$ is perfect)

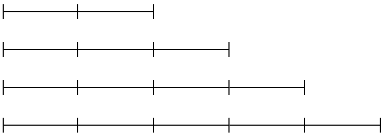
Definitions from the Elements, Book VII

An unit is that by virtue of which each of the things that exist is called one.

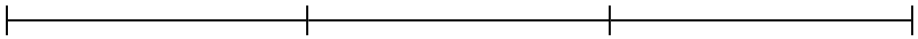
1 

A number is a multitude composed of units.

2
3
4
5



A number is a part of a number, the less of the greater, when it measures the greater.

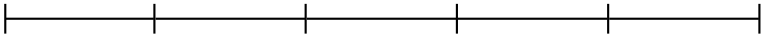


2 is part of 6

Definitions from the Elements, Book VII

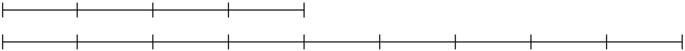
A prime number is that which is measured by an unit alone.

5 is prime:



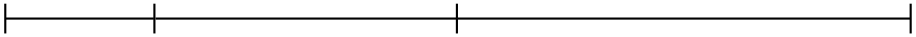
Numbers prime to one another are those which are measured by an unit alone as a common measure.

4 and 9 are prime to each other:



A perfect number is that which is equal to its own parts.

$6 = 1 + 2 + 3$:



Book VII, Proposition 1

Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another.

What does this say? Let's look at an example...

Book VII, Proposition 1: example

x	y	action
140	33	subtract: $140 - 33 = 107$

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140	33	subtract: $140 - 33 = 107$
107	33	subtract: $107 - 33 = 74$

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x	y	action
140	33	subtract: $140 - 33 = 107$
107	33	subtract: $107 - 33 = 74$
74	33	subtract: $74 - 33 = 41$

Book VII, Proposition 1: example

x	y	action
140	33	subtract: $140 - 33 = 107$
107	33	subtract: $107 - 33 = 74$
74	33	subtract: $74 - 33 = 41$
41	33	subtract: $41 - 33 = 8$

Book VII, Proposition 1: example

x	y	action
140	33	subtract: $140 - 33 = 107$
107	33	subtract: $107 - 33 = 74$
74	33	subtract: $74 - 33 = 41$
41	33	subtract: $41 - 33 = 8$
8	33	swap

Book VII, Proposition 1: example

x	y	action
140	33	subtract: $140 - 33 = 107$
107	33	subtract: $107 - 33 = 74$
74	33	subtract: $74 - 33 = 41$
41	33	subtract: $41 - 33 = 8$
8	33	swap
33	8	subtract: $33 - 8 = 25$

Book VII, Proposition 1: example

x	y	action
140	33	subtract: $140 - 33 = 107$
107	33	subtract: $107 - 33 = 74$
74	33	subtract: $74 - 33 = 41$
41	33	subtract: $41 - 33 = 8$
8	33	swap
33	8	subtract: $33 - 8 = 25$
25	8	subtract: $25 - 8 = 17$

Book VII, Proposition 1: example

x	y	action
140	33	subtract: $140 - 33 = 107$
107	33	subtract: $107 - 33 = 74$
74	33	subtract: $74 - 33 = 41$
41	33	subtract: $41 - 33 = 8$
8	33	swap
33	8	subtract: $33 - 8 = 25$
25	8	subtract: $25 - 8 = 17$
17	8	subtract: $17 - 8 = 9$

Book VII, Proposition 1: example

x	y	action
140	33	subtract: $140 - 33 = 107$
107	33	subtract: $107 - 33 = 74$
74	33	subtract: $74 - 33 = 41$
41	33	subtract: $41 - 33 = 8$
8	33	swap
33	8	subtract: $33 - 8 = 25$
25	8	subtract: $25 - 8 = 17$
17	8	subtract: $17 - 8 = 9$
9	8	subtract: $9 - 8 = 1$

Euclid and the Greatest Common Divisor

Book VII, Proposition 1: example

x	y	action
140	33	subtract: $140 - 33 = 107$
107	33	subtract: $107 - 33 = 74$
74	33	subtract: $74 - 33 = 41$
41	33	subtract: $41 - 33 = 8$
8	33	swap
33	8	subtract: $33 - 8 = 25$
25	8	subtract: $25 - 8 = 17$
17	8	subtract: $17 - 8 = 9$
9	8	subtract: $9 - 8 = 1$
1	8	stop: 140 and 33 are prime to each other

Greatest Common Divisor (Measure)

- Given two positive whole numbers, say 1035 and 759
- We are looking for a common divisor (measure)
  - $1035 = 3 \cdot 345$ and $759 = 3 \cdot 253$ so 3 is a common divisor

Greatest Common Divisor (Measure)

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- We are looking for a common divisor (measure)
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    - $1035 = 23 \cdot 45$ and $759 = 23 \cdot 33$ so 23 is a common divisor
- We want the greatest common divisor, however. Is it 23?

Greatest Common Divisor (Measure)

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  - $1035 = 23 \cdot 45$ and $759 = 23 \cdot 33$ so 23 is a common divisor
- We want the greatest common divisor, however. Is it 23?
- No.
  $1035 = 69 \cdot 15$ and $759 = 69 \cdot 11$, so 69 is a common divisor.
  Is it the greatest common divisor?

Greatest Common Divisor (Measure)

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- We are looking for a common divisor (measure)
  - $1035 = 3 \cdot 345$ and $759 = 3 \cdot 253$ so 3 is a common divisor
  - $1035 = 23 \cdot 45$ and $759 = 23 \cdot 33$ so 23 is a common divisor
- We want the greatest common divisor, however. Is it 23?
- No.
  $1035 = 69 \cdot 15$ and $759 = 69 \cdot 11$, so 69 is a common divisor.
  Is it the greatest common divisor?
- Yes.
  $1035 = 3 \cdot 3 \cdot 5 \cdot 23$ and $759 = 3 \cdot 11 \cdot 23$

> We want a method to determine the greatest common divisor of any pair (a, b) of whole numbers and we don't want to work "too hard."

Method 1: brute force (ignoring Euclid)

- Let m be the smaller of a and b
- Try all candidate divisors $m, m-1, m-2, \ldots, 1$
- For each candidate, check if it is a common divisor
- Stop when a common divisor has been found: this is the greatest one

For large values of a and b, this is very labor-intensive! We can do much, much better.

Euclid's idea

Euclid's (simplified) rule

Suppose x and y are positive integers with $x \geq y$. Then $\gcd(x, y) = \gcd(x - y, y)$.

Proof sketch

- $\gcd(x, y) \leq \gcd(x - y, y)$
- $\gcd(x - y, y) \leq \gcd(x, y)$

Euclid's idea: proof details, part 1

Show $\gcd(x, y) \leq \gcd(x - y, y)$

Let d be a common divisor of x and y.

We need to show that $d \mid (x - y)$ and $d \mid y$.

Euclid's idea: proof details, part 1

Show $\gcd(x, y) \leq \gcd(x - y, y)$

Let d be a common divisor of x and y.

We need to show that $d \mid (x - y)$ and $d \mid y$.

Since $d \mid x$ and $d \mid y$, we can write $x = dq_1$ and $y = dq_2$. Then,

$$\begin{aligned} x - y &= dq_1 - dq_2 \\ &= d(q_1 - q_2) \\ &= dq_3 \end{aligned}$$

In other words, $d \mid (x - y)$.

Euclid's idea: proof details, part 2

Show $\gcd(x - y, y) \leq \gcd(x, y)$

Let d be a common divisor of $x - y$ and y.

We need to show that $d \mid x$ and $d \mid y$.

Euclid's idea: proof details, part 2

Show $\gcd(x-y, y) \leq \gcd(x, y)$

Let d be a common divisor of $x - y$ and y.

We need to show that $d \mid x$ and $d \mid y$.

Since $d \mid (x-y)$ and $d \mid y$, we can write $x - y = dq_1$ and $y = dq_2$. Then,

$$\begin{aligned} x &= (x-y) + y \\ &= dq_1 + dq_2 \\ &= d(q_1 + q_2) \\ &= dq_3 \end{aligned}$$

In other words, $d \mid x$.

Method 2: repeated subtraction

Euclid's simplified rule: if $x \geq y$ then $\gcd(x, y) = \gcd(x - y, y)$

x	y	action
1035	759	subtract: $1035 - 759 = 276$

Method 2: repeated subtraction

Euclid's simplified rule: if $x \geq y$ then $\gcd(x, y) = \gcd(x - y, y)$

x	y	action
1035	759	subtract: $1035 - 759 = 276$
276	759	swap

Method 2: repeated subtraction

Euclid's simplified rule: if $x \geq y$ then $\gcd(x, y) = \gcd(x - y, y)$

x	y	action
1035	759	subtract: $1035 - 759 = 276$
276	759	swap
759	276	subtract: $759 - 276 = 483$

Method 2: repeated subtraction

Euclid's simplified rule: if $x \geq y$ then $\gcd(x, y) = \gcd(x - y, y)$

x	y	action
1035	759	subtract: $1035 - 759 = 276$
276	759	swap
759	276	subtract: $759 - 276 = 483$
483	276	subtract: $483 - 276 = 207$

Method 2: repeated subtraction

Euclid's simplified rule: if $x \geq y$ then $\gcd(x, y) = \gcd(x - y, y)$

x	y	action
1035	759	subtract: $1035 - 759 = 276$
276	759	swap
759	276	subtract: $759 - 276 = 483$
483	276	subtract: $483 - 276 = 207$
207	276	swap

Method 2: repeated subtraction

Euclid's simplified rule: if $x \geq y$ then $\gcd(x, y) = \gcd(x - y, y)$

x	y	action
1035	759	subtract: $1035 - 759 = 276$
276	759	swap
759	276	subtract: $759 - 276 = 483$
483	276	subtract: $483 - 276 = 207$
207	276	swap
276	207	subtract: $276 - 207 = 69$

Method 2: repeated subtraction

Euclid's simplified rule: if $x \geq y$ then $\gcd(x, y) = \gcd(x - y, y)$

x	y	action
1035	759	subtract: $1035 - 759 = 276$
276	759	swap
759	276	subtract: $759 - 276 = 483$
483	276	subtract: $483 - 276 = 207$
207	276	swap
276	207	subtract: $276 - 207 = 69$
69	207	swap

Method 2: repeated subtraction

Euclid's simplified rule: if $x \geq y$ then $\gcd(x, y) = \gcd(x - y, y)$

x	y	action
1035	759	subtract: $1035 - 759 = 276$
276	759	swap
759	276	subtract: $759 - 276 = 483$
483	276	subtract: $483 - 276 = 207$
207	276	swap
276	207	subtract: $276 - 207 = 69$
69	207	swap
207	69	subtract: $207 - 69 = 138$

Method 2: repeated subtraction

Euclid's simplified rule: if $x \geq y$ then $\gcd(x, y) = \gcd(x - y, y)$

x	y	action
1035	759	subtract: $1035 - 759 = 276$
276	759	swap
759	276	subtract: $759 - 276 = 483$
483	276	subtract: $483 - 276 = 207$
207	276	swap
276	207	subtract: $276 - 207 = 69$
69	207	swap
207	69	subtract: $207 - 69 = 138$
138	69	subtract: $138 - 69 = 69$

Method 2: repeated subtraction

Euclid's simplified rule: if $x \geq y$ then $\gcd(x, y) = \gcd(x - y, y)$

x	y	action
1035	759	subtract: $1035 - 759 = 276$
276	759	swap
759	276	subtract: $759 - 276 = 483$
483	276	subtract: $483 - 276 = 207$
207	276	swap
276	207	subtract: $276 - 207 = 69$
69	207	swap
207	69	subtract: $207 - 69 = 138$
138	69	subtract: $138 - 69 = 69$
69	69	subtract: $69 - 69 = 0$

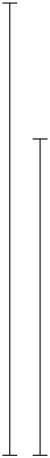
Method 2: repeated subtraction

Euclid's simplified rule: if $x \geq y$ then $\gcd(x, y) = \gcd(x - y, y)$

x	y	action
1035	759	subtract: $1035 - 759 = 276$
276	759	swap
759	276	subtract: $759 - 276 = 483$
483	276	subtract: $483 - 276 = 207$
207	276	swap
276	207	subtract: $276 - 207 = 69$
69	207	swap
207	69	subtract: $207 - 69 = 138$
138	69	subtract: $138 - 69 = 69$
69	69	subtract: $69 - 69 = 0$
0	69	stop: 69 is the greatest common divisor

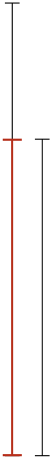
Method 2: as conceived by Euclid

Find gcd of 30 and 21



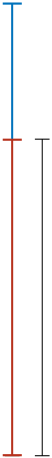
Method 2: as conceived by Euclid

Find gcd of 30 and 21



Method 2: as conceived by Euclid

Find gcd of 30 and 21



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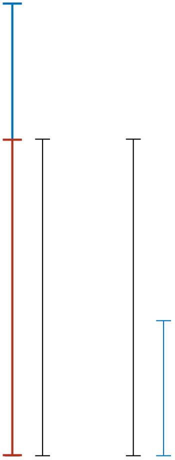

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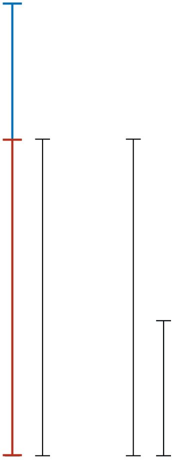

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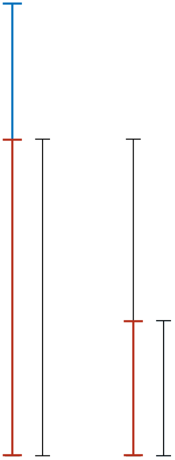

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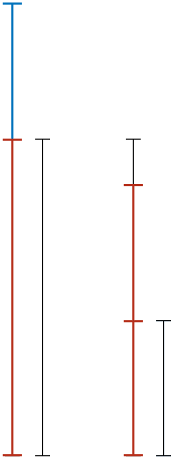

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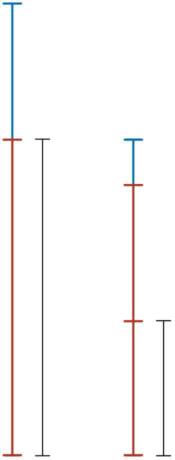

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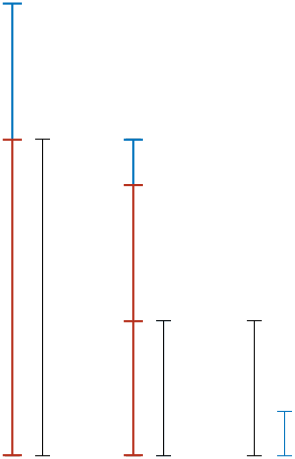

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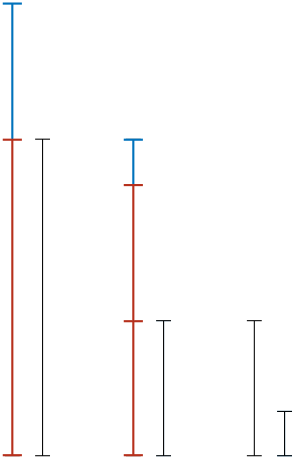

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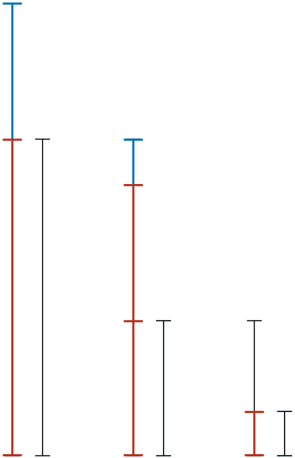

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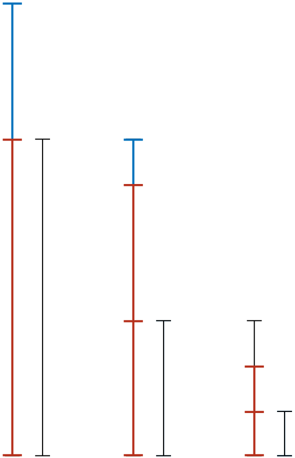

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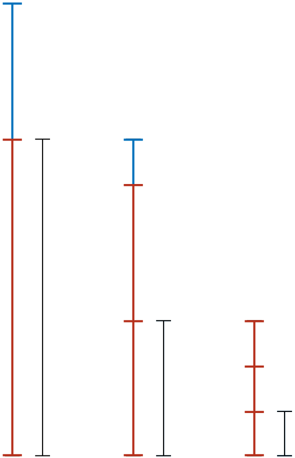

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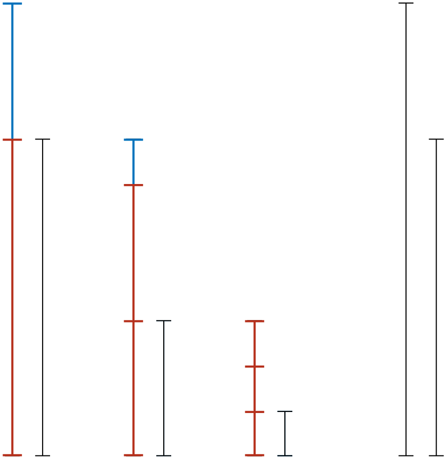

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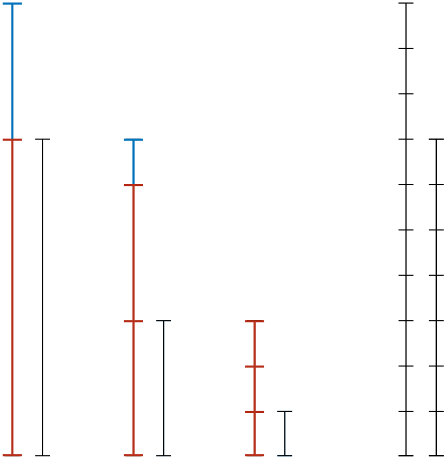

Euclid's idea, using division

Key idea

Repeated subtraction is just (fourth grade) division!

Euclid's rule

If x and y are positive integers with $x \geq y$, then
$\gcd(x, y) = \gcd(x \bmod y, y) = \gcd(y, x \bmod y)$.

Method 3: Euclid's method

Euclid's rule: if $x \geq y$ then $\gcd(x, y) = \gcd(y, x \bmod y)$

x	y	action
1035	759	divide: $1035 = 1 \cdot 759 + 276$

Method 3: Euclid's method

Euclid's rule: if $x \geq y$ then $\gcd(x, y) = \gcd(y, x \bmod y)$

x	y	action
1035	759	divide: $1035 = 1 \cdot 759 + 276$
759	276	divide: $759 = 2 \cdot 276 + 207$

Method 3: Euclid's method

Euclid's rule: if $x \geq y$ then $\gcd(x, y) = \gcd(y, x \bmod y)$

x	y	action
1035	759	divide: $1035 = 1 \cdot 759 + 276$
759	276	divide: $759 = 2 \cdot 276 + 207$
276	207	divide: $276 = 1 \cdot 207 + 69$

Method 3: Euclid's method

Euclid's rule: if $x \geq y$ then $\gcd(x, y) = \gcd(y, x \bmod y)$

x	y	action
1035	759	divide: $1035 = 1 \cdot 759 + 276$
759	276	divide: $759 = 2 \cdot 276 + 207$
276	207	divide: $276 = 1 \cdot 207 + 69$
207	69	divide: $207 = 3 \cdot 69 + 0$

Method 3: Euclid's method

Euclid's rule: if $x \geq y$ then $\gcd(x, y) = \gcd(y, x \bmod y)$

x	y	action
1035	759	divide: $1035 = 1 \cdot 759 + 276$
759	276	divide: $759 = 2 \cdot 276 + 207$
276	207	divide: $276 = 1 \cdot 207 + 69$
207	69	divide: $207 = 3 \cdot 69 + 0$
69	0	stop: 69 is the greatest common divisor

Euclid's method

We might call Euclid's method the granddaddy of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day.

Donald Knuth
The Art of Computer Programming

Java implementation

Euclid's rule

If x and y are positive integers with $x \geq y$, then $\gcd(x, y) = \gcd(y, x \bmod y)$.

```
public static int gcd(int x, int y)
{
    if (y == 0)
        return x;
    else
        return gcd(y, x % y);
}
```

Efficiency of Euclid's algorithm

Gabriel Lamé
1795 – 1870
(French; not Greek, not ancient!)

Lamé's theorem

To find the greatest common divisor of integers x and y using Euclid's algorithm takes at most $5k$ steps, where k is the number of digits of y.

An extension of Euclid's algorithm

In addition to finding $\gcd(x, y) = d$, we might want values a and b such that

$$ax + by = d$$

We know $\gcd(1035, 759) = 69$. In addition,

$$3 \cdot 1035 + (-4) \cdot 759 = 69$$

A small modification to Euclid's method determines these two values.

Euclid contributes to the Internet age

- Public-key cryptography: how to keep a secret, yet still communicate?
- Two players, traditionally known as "Alice" and "Bob"
- Bob:
  - chooses two large prime numbers, p and q
  - computes a public key that everyone can know. This key includes the product pq, but not the two primes.
  - computes a private key, computed with an extended version of Euclid's algorithm
- Alice:
  - encodes a message, using Bob's public key
- Bob decodes the message using his private key

Number theory, once thought to be an abstract area of mathematics without application, is anything but. **Hats off to Euclid!**